An alternative method to measure the likelihood of a financial crisis in an emerging market

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Abstract

This paper utilizes an early warning system in order to measure the likelihood of a financial crisis in an emerging market economy. We introduce a methodology, where we can both obtain a likelihood series and analyze the time-varying effects of several macroeconomic variables on this likelihood. Since the issue is analyzed in a non-linear state space framework, the extended Kalman filter emerges as the optimal estimation algorithm.

Taking the Turkish economy as our laboratory, the results indicate that both the derived likelihood measure and the estimated time-varying parameters are meaningful and can successfully explain the path that the Turkish economy had followed between 2000 and 2006. The estimated parameters also suggest that overvalued domestic currency, current account deficit and the increase in the default risk increase the likelihood of having an economic crisis in the economy. Overall, the findings in this paper suggest that the estimation methodology introduced in this paper can also be applied to other emerging market economies as well.

Keywords: Extended Kalman filter; Financial crises; Emerging markets

1. Introduction

The last decade witnessed an increased frequency of crises in the emerging market economies, which necessitated a deeper research on the macroeconomic dynamics of these financial crises. Although the causes and the consequences of these financial crises are studied to a significant extent, the literature could still be expanded, especially on introducing alternative methodologies to predict the likelihood of financial crises in emerging markets.¹ Given the empirical fact that there is contagion in terms of crises between the emerging markets, the vitality of having an appropriate likelihood measure of having a financial crises will be much clearer.

Based on the above discussion, the goal of this paper is to utilize an early warning system in order to measure the likelihood of a financial crisis in the economy. It is not an attempt to formulate or test specific

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¹Refs. [1–4] are some of the studies, which investigate the sources and the consequences of financial crises in emerging markets in details.

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theories, but rather to obtain a likelihood measure. The Turkish economy, which reflects the identifying characteristics of an emerging market, is taken as our case study.

Thus, this paper mainly contributes to the literature by applying a mechanical statistical tool to the field of financial economics. We handle the issue within a system of equations, where the likelihood of having a financial crisis is treated as an unobserved state variable to be estimated. Moreover, we allow the system parameters to vary over time, where the volatile nature of the emerging market economies and the unstable relation between the macroeconomic variables can be nicely captured. These two characteristics of the model induce us to employ a non-linear state space model, where the unobserved state variables and the time-varying parameters, which appear in the model in a non-linear form, can be simultaneously estimated. In this case, the extended Kalman filter (EKF), which is discussed by Chen [5] in details, emerges as the appropriate estimation algorithm. As implied above, such an approach has not been employed to study the dynamics of financial economics in emerging market economies.

After estimating the above-mentioned system of equations, we obtain two distinct sets of results. First, we derive a time series, which indicates the likelihood of having a financial crisis in the economy. Using this estimated series, we assess how this likelihood measure has evolved over time and whether our model is capable of predicting the financial crisis that the Turkish economy had experienced in the last decade. Second, and importantly, we track down the time-varying effects of several macroeconomic variables on this likelihood measure. Given the projections about the macroeconomic variables of interest, such an exercise will allow us to make predictions about the likelihood of having a financial crisis in the economy.

The rest of the paper is organized as follows. Section 2 summarizes both the literature on the dynamics of financial crises in emerging markets and the recent development in the Turkish economy. While Section 3 presents both the model and the estimation procedure, Section 4 provides the results with brief interpretation. The final section concludes.

2. Literature review and the Turkish economy

Before 1990s, currency crises were thought to have a significant predictable component, as was captured in a series of first-generation models which assert that a fixed exchange rate policy combined with excessively expansionary pre-crisis fundamentals push the economy into crisis, with the private sector trying to profit from dismantling the inconsistent policies. However, the currency crises experienced throughout the European Monetary System period in spite of the sound macroeconomic fundamentals challenged this view.

More recently, the literature on capital inflows and capital inflow problems has suggested another potential source of instability. Sudden reversals of capital flows may cause liquidity crises as happened in debt crises of 1982, the Mexican crisis in 1994 and the Asian crises in 1997–1998. Thus, the liberalization of capital account transactions by allowing this type of short-term capital flows may contribute to the instability of the flow of reserves and inability of the country to the peg the domestic currency.

After it was perceived that the currency crises of the 1990s had a contagious effect, especially in the emerging markets, the literature on estimating the probability of a financial crisis picked up. Among these studies, the leading indicators approach by Kaminsky et al. [9], the weak fundamentals approach by Sachs et al. [10] and the probit regression model by Frankel and Rose [11] deserve special attention.

On the other hand, one strand of the literature investigates the structure of the financial markets. Disyatat [12] shows that countries with higher degrees of financial market imperfection are more likely to suffer from a contraction in the wake of a currency crisis. Such a result is consistent with several earlier studies such as Williamson [13], Bernanke and Gertler [14], Agenor and Aizenman [15], and Holmstrom and Tirole [16], all of which stress on the role of credit market imperfections.

The literature review on the financial crises can easily be extended. However, the main focus of this study is to propose an alternative approach to obtain a likelihood measure of a financial crisis and most of the studies that are not discussed within these lines use conventional estimation techniques to handle the problem. Thus, we prefer to mention only the most cited studies and devote more effort to introducing our estimation

\[\text{Refs. [6,7] are two of the most cited papers in this context.}\]
\[\text{See Ref. [8] for a detailed analysis of the issue.}\]
methodology, which have not been employed in any of the previous studies in the literature. However, before that, it is worth presenting the macroeconomic developments in the Turkish economy during the sample period.

2.1. Turkish economy: 2000–2006

Similar to other emerging markets, the Turkish economy can be characterized by having an unsustainable output growth performance, high inflationary environment and shallow financial markets, which increases the fragility of the financial system to economic developments. Combining these characteristics with the Russian crisis and the two devastating earthquakes of 1999, it was clear that the Turkish economy had to adopt a stabilization program, which would promote sustainable output growth and low levels of inflation. The main aims of the program were tight fiscal and monetary policies, structural reforms and the use of a pre-determined exchange rate path as a nominal anchor. However, although the program seemed to achieve its goals during the initial phases, huge current account deficits due to an over-valued domestic currency and the open positions of the commercial banks in the following periods increased the doubts about the success of the program. As a result, the outflow of foreign funds from Turkey and the sharp increase in T-bill rates led to financing difficulties by some private and state banks. The subsequent November 2000 crisis led to a significant erosion of the capital base of the banking sector and revealed the fragility of the system. The loss of credibility of the exchange rate regime and finally the abolition of the exchange rate peg in February 2001 further hit the already weak banking sector. Central Bank was forced to quit the exchange rate peg and started to implement floating exchange rate regime.

In order to reduce the detrimental effects of the financial crisis, the government immediately adopted a new program “Transition to a Strong Economy”, where the strategy was strongly based on market-orientation and openness to the world economy. The heavy emphasis was put on the implementation of the key structural reforms in public finance and banking sector to place the Turkish economy on a sustainable growth path. In addition, since the second half of 2001, the Central Bank has been implementing an implicit inflation targeting regime and using the overnight interest rates as the policy instrument in a forward-looking manner to achieve the inflation targets.

Consequently, after its launch, the program seems to produce some desirable outcomes, especially on the inflation dynamics. Following the adoption of the floating exchange rate regime, the Central Bank was successful in bringing down the inflation to single digits. In addition, the government could achieve a sort of fiscal discipline, which eased the problems on debt sustainability. However, it is still early to claim that these positive developments resulted in a sustainable macroeconomic path. High current account deficits and the susceptibility of the financial markets to external shocks still emerge as two important problems, which should definitely be solved to attain a sustainable path. To give an example, during the May–June global financial turbulence in 2006, Turkey was the most negatively affected economy among the emerging markets, possibly because of its overvalued domestic currency and huge current account deficit. In fact, the next section will introduce a framework, which will enable us to see how the above-mentioned factors affected the likelihood of having another financial crisis.

3. The model and the estimation procedure

In this section, first, we briefly introduce the equations in the system and discuss the economic implications behind. The system consists of the following equations:

\[ D_{t+1} = \gamma_0,D_t + \gamma_1,PSBR_t + \gamma_2,DEPR_t + \epsilon_{1,t}, \]
\[ \left( \frac{1 + R_t}{1 + DEPR_t} \right)_{t+1} = \alpha_0,\left( \frac{1 + R_t}{1 + DEPR_t} \right)_t + \alpha_1,DEPR_t + \alpha_2,EMBI_t + \epsilon_{2,t}, \]
\[ MIS_{t+1} = \beta_0,MIS_t + \beta_1,\left( \frac{1 + R_t}{1 + DEPR_t} \right)_t + \epsilon_{3,t}, \]
\[ CA_{t+1} = \psi_{0,t} CA_t + \psi_{1,t} RER_t + \psi_{2,t} Y_t + \varepsilon_{4,t}, \]  
\[ Pr_{t+1} = \phi_{0,t} Pr_t + \phi_{1,t} CA_t + \phi_{2,t} \left( \frac{1 + R_t}{1 + DEPR_t} \right)_t + \phi_{3,t} MIS_t + \phi_{4,t} EMBI_t + \varepsilon_{5,t}. \]

In the above system, \( D_t \) is the domestic debt to GDP ratio, \( PSBR_t \) is the public sector borrowing requirement, \( DEPR_t \) is the depreciation rate of the domestic currency (TL/USD), the ratio of \((1 + R_t)/(1 + DEPR_t)\) is defined as monetary spread where \( R_t \) is the Central Bank’s interbank interest rate, \( EMBI_t \) is the emerging market bond index, \( MIS_t \) is the exchange rate misalignment which is defined as deviation of real exchange rate from its long run trend, \( CA_t \), is the current account deficit ratio to GDP, \( RER_t \) is the real exchange rate, \( Y_t \) is the nominal GDP and \( Pr_t \) is the unobserved risk premium.\(^4\)

The first equation attempts to summarize the debt dynamics of the Turkish economy: It states that the domestic debt to GDP ratio depends on its own lagged value, public sector borrowing requirement and the depreciation rate. An increase in the public sector borrowing requirement raises the current period domestic debt burden and the depreciation of currency increases the foreign exchange denominated debt. Therefore, these two factors are expected to affect the debt stock to GDP ratio negatively.

The spread between the Central Bank’s interbank interest rate and the depreciation rate of the domestic currency is used as an indicator of the monetary policy where the monetary spread defined as \((1 + R_t)/(1 + DEPR_t)\), consistent with Berument [17]. The second equation states that the monetary spread depends on its own lag, the domestic debt to GDP ratio of the economy and the EMBI spread. We expect that an increase in the domestic debt to GDP ratio and the EMBI spread raise the above-defined monetary spread.

The third equation shows that the misalignment depends on its own lag and the monetary spread. We expect that an increase in the monetary spread will increase the misalignment.

The fourth equation summarizes the current account dynamics of the Turkish economy simply: It states that the current account deficit to GDP ratio of the economy depends on its own lag, real exchange rate and the output. An increase in the real exchange rate and output raise the current account deficit and the current account to GDP ratio of the economy, respectively. In other words, appreciation of the exchange rate and positive output are expected to generate a current account deficit.

The last equation shows the specification for the unobserved variable, which indicates the likelihood of having a financial crisis in the economy. It is assumed that, the unobserved likelihood variable is affected by its own lagged value, the current account deficit to GDP ratio, the monetary spread, exchange rate misalignment and the EMBI spread. Since the current account ratio is of the main foot of external sector, deterioration in current account is expected to increase the likelihood of a currency crisis. As discussed in Ref. [18], if a current account deficit is accompanied by a real exchange rate appreciation, then the likelihood of a currency crisis is increased. In addition, consistent with the theoretical literature, it is expected that an increase in the monetary spread, misalignment and the EMBI spread increase the probability of a financial crisis.

Finally, it should be mentioned that the system parameters are assumed to vary over time. As discussed before, such an assumption allows us to capture the changing relationships between the macroeconomic variables over time, which is almost a stylized fact in the emerging markets literature. Thus, the above-mentioned system has 16 more equations, each of which describes a random walk process for the time-varying parameters. Since the changes in the relationships between the macroeconomic variables cannot be known a priori, it is convenient to assume a random walk specification for the parameters in the model.

3.1. Estimation procedure

This part introduces the state space representation of the model and the non-linearity, which necessitates the employment of EKF.

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\(^4\)The data are obtained from the Research Department of the Central Bank of the Republic of Turkey. The sample period is 2000:01–2006:12.
A linear state space form will have the following state and observation equations:

\[ X_{t+1} = \Phi X_t + \Psi K_t + Gw_t, \]

\[ y_t = HX_t + v_t. \]

In the linear form, the matrices \( \Phi, \Psi \) do not vary over time. Then, both the state variables and the parameters are estimated via standard Kalman filter. However, as mentioned above, the parameters in the model are time varying, which make the matrices \( \Phi \) and \( \Psi \) time dependent. Furthermore, since these matrices and the state vector \( X_t \), which includes the unobserved state variable to be estimated, appear in multiplicative forms, the state space model takes on a non-linear characteristic. The EKF is one of the most popular estimation techniques largely investigated for state estimation of such non-linear systems. It consists of using the standard Kalman filter equations to the first-order approximation of the non-linear model about the last estimate. The next section describes the non-linearity and the estimation procedure in details.

3.2. Applying EKF

The previous section stated that the model, which had the time-varying parameters and the unobserved state variables in multiplicative form, had a non-linear characteristic. To see that more clearly, let \( \Omega_t = \left[ x_{0,t}, x_{1,t}, \beta_{0,t}, \beta_{1,t}, \gamma_{0,t}, \gamma_{1,t}, \gamma_{2,t}, \varphi_{0,t}, \varphi_{1,t}, \varphi_{2,t}, \theta_{0,t}, \theta_{1,t}, \theta_{2,t}, \varphi_{0,t}, \varphi_{1,t}, \varphi_{2,t}, \phi_{0,t}, \phi_{1,t}, \phi_{2,t}, \phi_{3,t}, \phi_{4,t} \right] \) be the parameter vector to be estimated. Then Eqs. (6) and (7) should be treated as \( \Phi(\Omega_t), \Psi(\Omega_t), G(\Omega_t), H(\Omega_t) \), which are all functions of the parameter vector, \( \Omega_t \). In this case, the model becomes:

\[ X_{t+1} = \Phi(\Omega_t)X_t + \Psi(\Omega_t)K_t + G(\Omega_t)w_t, \]

\[ y_t = H(\Omega_t)X_t + v_t. \]

It can be seen that Eqs. (8) and (9) are in the non-linear form. Since we assume a random walk process for all of the parameters in the model, we have

\[ \Omega_{t+1} = \Omega_t + \zeta_{t+1}, \]

where \( \zeta_t \) is any zero-mean white noise sequence uncorrelated with \( v_t \) and with pre-assigned positive definite variances \( \text{Var}(\zeta_t) = S_t \). If we treat the above equations as the new state vector and combine them, the new state space model becomes:

\[ \begin{bmatrix} x_{t+1} \\ \Omega_{t+1} \end{bmatrix} = \begin{bmatrix} \Phi(\Omega_t)x_t \\ \Omega_t \end{bmatrix} + \begin{bmatrix} \Psi(\Omega_t)K_t + G(\Omega_t)\zeta_t \\ \zeta_t \end{bmatrix}, \]

\[ y_t = [H(\Omega_t)0] \begin{bmatrix} x_t \\ \Omega_t \end{bmatrix} + \eta_t. \]

The non-linearity can also be seen in Eq. (11). Then EKF procedure can be applied to estimate the state vector, which contains \( \Omega_t \) as one of its components. That is, \( \Omega_t \) is estimated optimally in an adaptive way. This procedure is called adaptive system identification, as noted in Ref. [19]. Both the EKF algorithm and its application in non-linear state space models are discussed in Refs. [20,5] in details. In addition, the algorithm, which is also used in this model, are presented in Appendix A.

4. Estimation results and discussion

As it is mentioned in the previous section, the EKF is the appropriate estimation algorithm in non-linear state space models. Since the paper estimates both the time-varying parameters of the system and constructs a time series for the “likelihood of a financial crisis”, the state space framework turns out to have a non-linear
nature. Therefore, after the system of equations is written in a state space model, the EKF is executed and the parameters as well as the unobserved “crisis” measure are obtained.

Since the main goal of the study is to come up with a measure of crisis likelihood, this section is particularly devoted to the interpretation of the constructed series. Also, the estimated parameters in the “crisis” equation will be briefly discussed.5

4.1. The estimated “likelihood” of a financial crisis

Fig. 1 displays the unobserved “likelihood of a financial crisis” series, which steadily increases in the first phase of the sample and reaches a maximum on October 2001. After then, we detect a decline until the second quarter of 2002. Although the series tends to have a slight increase from May 2002 to November 2002, we observe a significant decrease until then. It should be reminded that the sample period is between January 2000 and December 2006.6 Such a time span includes the exchange rate based stabilization program, the financial crisis period and the first phase of the new economic program, which was intended for a structural transformation in the economy. Thus, the estimated ‘crisis’ series should be interpreted with respect to these important developments in the Turkish economy.

As implied before, following the 1998 Russian crisis and the devastating 1999 earthquake, it was clear that the Turkish economy entered into an unsustainable path, which necessitated the “2000–2002 program”. The program can be regarded as an exchange rate based stabilization program, where there were also attempts to reduce the debt burden with a tight fiscal policy. However, due to the reluctance in implementing the structural reforms and the fragile banking sector, it was soon anticipated by the agents that the program was far from being successful to attain the goals of low inflation and sustainable debt stock. Actually, the estimated “crisis” series indicate that the likelihood of having a financial crisis steadily increased even when the “2000–2002 program” was introduced. Finally, when the uncertainties regarding the success of the economic program aggravated, the Central Bank of the Republic of Turkey announced in February 2001 that it would no longer defend the exchange rate target and the monetary policy framework would operate within a floating exchange rate regime. In addition, beginning with May 2001, a new program, supported by the IMF, was announced.

5The other estimated parameters are available on request.

6Although we started the estimation process at mid 1999, due to the excessive fluctuations at the initial phase, which is a characteristic of recursive algorithms, we report the estimated series beginning from January 2000.
It can also be seen that the series did not reach a maximum at the collapse of the exchange rate based stabilization program but increased even after then. Such a result is totally consistent when the debt dynamics are taken into account. After the duty losses of the banking sector have been added to the debt stock of the government, the risk perceptions in the economy have further increased. The real interest rates have jumped up to around 30 percent. These negative developments on the fiscal side have brought the possibility that the government would default on its debt. As a result, the likelihood of having a further economic crisis has increased until the beginning of the fourth quarter of 2001. Such a finding suggests that even if February 2001 was regarded as the date of the financial crisis, the following months should be viewed as harder times.

At the end of 2001, the investors in the economy realized the first positive signs of the new stabilization program. Also, the government made it clear that it did not have any incentives to default on its debt. These positive factors can explain the decrease in the estimated “crisis” series. However, it can be seen from Fig. 1 that the series began to increase again by May 2002. This is also expected, since beginning from the second quarter of 2002, there was a high degree of political instability with intensified discussion about an early election. At the same time, the worsening health condition of the prime minister was another factor for this increase in the series.

On November 2002, the political instability came to a halt after a single party came into power. After then, the series had a downward trend for a short period of time, which stopped with the beginning of the Iraq war. At the second half of 2003, there is a steady decrease in the series, possibly due to the fiscal discipline that has been maintained. However, with the beginning of the tight monetary policy era by the FED in 2004, we again see a slight increase in the estimated likelihood series. Such a development is caused by the increase in the EMBI spread for the same period, which is assumed as one of the main determinants of the estimated likelihood series in the fifth equation.

Beginning from the second half of 2004, we see the sharpest decline in the series because of the positive changes in both internal and external factors. This decreasing trend was slightly reversed, beginning from May 2006. As mentioned above, the global financial turbulence for this particular period emerges as the main factor behind this increase.

As a final exercise, it may provide insightful results to check whether the estimated “crisis” series tend to co-move with the EMBI (emerging market bond index) spread. Prepared by J.P. Morgan for each emerging market economy, this spread is broadly used as an overall risk measure and reflects mainly the default risk for that particular market. In fact, the correlation between the two series is found to be 0.76, which further validates this close link.

4.2. Estimated time-varying parameters

After investigating the estimated crisis variable, the next step is to discuss the parameters regarding the variables in the crisis equation. First, as a common finding, it has been found that the parameters do not vary over time during the time span. Such a result indicates that each variable in the crisis equation has the same impact through time. It has been found that the end-sample estimate of the coefficient for the lagged value of the crisis takes a value of 0.8. Such a finding implies inertia such that once the economy is likely to suffer from a crisis; the situation tends to get worse unless some dramatic steps are taken.

On the other hand, the estimated coefficient for the current account balance is $-0.4$, which implies that if the economy generates a current account deficit, mostly due to an overvalued domestic currency, then the likelihood of having a crisis increases. Such a finding fits well with the experiences of the February 2001 crisis.

The coefficient for the real rate of return for the foreign investor is found to be $0.5$. Such a finding has important implications, especially in terms of the short-term capital flows. If the real rate of return on Turkish securities increases, then a short-term capital inflow is expected. Once the economy gets surrounded with uncertainty, then one would expect massive short-term capital outflows, which generally triggers a currency crisis. The estimated coefficient for the real rate of return provides support for this channel.

Another related finding is the estimated coefficient for the misalignment measure. The coefficient is estimated to be around $0.6$, which means that as the domestic currency gets overvalued, the likelihood of having a crisis increases dramatically. Such a finding also explains why the exchange rate based stabilization programs tend to be unsuccessful in emerging market economies.
Consequently, all of the estimated parameters for the crisis equation make sense and provides support for the path that the Turkish economy had followed during the sample period.

5. Conclusion

This paper proposes a new framework to utilize an early warning system about the likelihood of a financial crisis. Although the study focuses on the Turkish economy, the introduced system of equations seems to be valid for the other emerging market economies as well. The likelihood of a financial crisis is treated as an unobserved variable and it is estimated within the context of a semi-structural macroeconomic model.

After the semi-structural model is put into a state space form, the EKF is executed. Since the time-varying parameters and the state variables are to be estimated simultaneously, the state space model will have a non-linear nature, where the standard Kalman filter is no longer appropriate. In such cases, the EKF serves as the optimal estimation methodology.

The results indicate that both the derived “crisis” measure and the estimated parameters are meaningful and can successfully explain the path that the Turkish economy had followed between 2000 and 2006. The estimated time-varying parameters also suggest that overvalued domestic currency, current account deficit and the increase in the default risk increase the likelihood of having an economic crisis in the economy. These results suggest that the estimation methodology introduced in this paper can also be applied to other emerging market economies as well.

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Appendix A. Extended Kalman filter algorithm

Following algorithm presents the results when the EKF is applied to the estimation of the parameter vector $\theta = (x_1, x_2, \rho, \gamma_1, x_3, \gamma_2, x_4, \gamma_3, \gamma_4)$. In addition to the usual Kalman filter algorithm, in here we have also random parameters, which are assumed to be evolving according to the random walk process. Simply, in the EKF case, because of the non-linear relationship, we linearize the process and measurement functions at the current state estimate by using the partial derivatives and then apply the usual Kalman filter algorithm.

The unobserved state vector $X$ can be seen as partitioned into two parts: one is the usual unobserved state variables and other is the unknown parameter vector:

$$X(t) = \begin{pmatrix} x(t) \\ \theta(t-1) \end{pmatrix}, \quad \tilde{K}(t) = \begin{pmatrix} K(t) \\ L(t) \end{pmatrix}, \quad \tilde{P}(t) = \begin{pmatrix} P_1(t) & P_2(t) \\ P_2^T(t) & P_3(t) \end{pmatrix},$$  \hspace{1cm} (A.1)

where $\tilde{K}$ and $\tilde{P}$ are the Kalman gain and covariance matrix for the extended state. Then the general algorithm:

$$S_t = H_tP_1(t)H_t^T + H_tP_2(t)D_t^T + D_tP_2^T(t)H_t^T + D_tP_3(t)D_t^T + R_2,$$ \hspace{1cm} (A.2)

$$L(t) = [P_2^T(t)H_{t-1}^T + P_3(t)D_t^T]S_t^{-1},$$ \hspace{1cm} (A.3)

$$K(t) = [F_tP_1(t)H_t^T + M_tP_2^T(t)H_t^T + F_tP_2(t)D_t^T + M_tP_3(t)D_t^T + R_{12}]S_t^{-1},$$ \hspace{1cm} (A.4)

$$P_1(t+1) = F_tP_1(t)F_t^T + F_tP_2(t)M_t^T + M_tP_2^T(t)F_t^T + M_tP_3(t)M_t^T - K(t)S_tK^T(t) + R_1,$$ \hspace{1cm} (A.5)

$$P_2(t+1) = F_tP_2(t) + M_tP_3(t) - K(t)S_tL_t^T(t),$$ \hspace{1cm} (A.6)

$$P_3(t+1) = P_3(t) - L(t)S_tL_t^T(t) + Q,$$ \hspace{1cm} (A.7)
\[ \dot{x}(t+1) = F_t \dot{x}(t) + G_t u(t) + K(t) \{ z(t) - H_t \dot{x}(t) \}, \] (A.8)

\[ \dot{\theta}(t) = \dot{\theta}(t-1) + L(t) [ z(t) - H_{t-1} \dot{x}(t) ] . \] (A.9)

Here

\[ F_t = F(\dot{\theta}(t)), \quad G_t = G(\dot{\theta}(t)), \quad H_t = H(\dot{\theta}(t)), \] (A.10)

\[ M_t = M(\dot{\theta}(t), \dot{x}(t), u(t)) \text{ with } M(\dot{\theta}, x, u) = \frac{\partial}{\partial \theta} [ F(\theta)x + G(\theta)u ] \bigg|_{\theta=\dot{\theta}} \] (A.11)

and

\[ D_t = D(\dot{\theta}(t-1), \dot{x}(t)) \text{ with } D(\dot{\theta}, x) = \frac{\partial}{\partial \theta} [ H(\theta)x ] \bigg|_{\theta=\dot{\theta}} . \] (A.12)

Finally, regarding the initial values:

\[ P_1(0) = P_{1,0}, \quad P_2(0) = 0, \quad P_3(0) = P_{3,0}, \quad \dot{x}(0) = x_0, \quad \dot{\theta}(0) = \theta_0. \] (A.13)

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